

# UNIQUENESS OF GIBBS MEASURE FOR MODELS WITH UNCOUNTABLE SET OF SPIN VALUES ON A CAYLEY TREE

YU. KH. ESHKABILOV, F. H. HAYDAROV, U. A. ROZIKOV

**ABSTRACT.** We consider models with nearest-neighbor interactions and with the set  $[0, 1]$  of spin values, on a Cayley tree of order  $k \geq 1$ . It is known that the "splitting Gibbs measures" of the model can be described by solutions of a nonlinear integral equation. For arbitrary  $k \geq 2$  we find a sufficient condition under which the integral equation has unique solution, hence under the condition the corresponding model has unique splitting Gibbs measure.

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## 1. INTRODUCTION

In this paper we consider models (Hamiltonians) with a nearest neighbor interaction and uncountably many spin values on a Cayley tree.

One of the central problems in the theory of Gibbs measures is to describe infinite-volume (or limiting) Gibbs measures corresponding to a given Hamiltonian. The existence of such measures for a wide class of Hamiltonians was established in the groundbreaking work of Dobrushin (see, e.g. [18]). However, a complete analysis of the set of limiting Gibbs measures for a specific Hamiltonian is often a difficult problem.

There are several papers devoted to models on Cayley trees, see for example [1]-[6], [8], [9], [12], [14]-[16], [19], [20], [22]. All these works devoted to models with a finite set of spin values. These models have the following common property: The existence of finitely many translation-invariant and uncountable numbers of the non-translation-invariant extreme Gibbs measures. Also for several models (see, for example, [5, 8, 15, 16]) it were proved that there exist three periodic Gibbs measures (which are invariant with respect to normal subgroups of finite index of the group representation of Cayley tree) and there are uncountable number of non-periodic Gibbs measures.

In [7] the Potts model with a countable set of spin values on a Cayley tree is considered and it was showed that the set of translation-invariant splitting Gibbs measures of the model contains at most one point, independently on parameters of the Potts model with countable set of spin values on Cayley tree. This is a crucial difference from the models with a finite set of spin values, since the last ones may have more than one translation-invariant Gibbs measures.

How "rich" is the set of translation-invariant Gibbs measures for models with an uncountable spin values? In [17] models with nearest-neighbor interactions and with the

set  $[0, 1]$  of spin values, on a Cayley tree of order  $k \geq 1$  are considered and we reduced the problem of describing the "splitting Gibbs measures" of the model to the description of the solutions of some nonlinear integral equation. For  $k = 1$  we showed that the integral equation has a unique solution. In case  $k \geq 2$  some models (with the set  $[0, 1]$  of spin values) which have a unique splitting Gibbs measure are constructed. In this paper we continue this investigations and give a sufficient condition on Hamiltonian of the model with an uncountable set of spin values under which the model has unique translation-invariant splitting Gibbs measure. But we have not any example of model (with uncountable spin values) with more than one translation-invariant Gibbs measure. So this is still an open problem to find such a model.

## 2. PRELIMINARIES

A Cayley tree  $\mathcal{G}^k = (V, L)$  of order  $k \geq 1$  is an infinite homogeneous tree (see [1]), i.e., a graph without cycles, with exactly  $k + 1$  edges incident to each vertices. Here  $V$  is the set of vertices and  $L$  that of edges (arcs).

Consider models where the spin takes values in the set  $[0, 1]$ , and is assigned to the vertexes of the tree. For  $A \subset V$  a configuration  $\sigma_A$  on  $A$  is an arbitrary function  $\sigma_A : A \rightarrow [0, 1]$ . Denote  $\Omega_A = [0, 1]^A$  the set of all configurations on  $A$ . A configuration  $\sigma$  on  $V$  is then defined as a function  $x \in V \mapsto \sigma(x) \in [0, 1]$ ; the set of all configurations is  $[0, 1]^V$ . The (formal) Hamiltonian of the model is :

$$H(\sigma) = -J \sum_{\langle x, y \rangle \in L} \xi_{\sigma(x)\sigma(y)}, \quad (2.1)$$

where  $J \in \mathbb{R} \setminus \{0\}$  and  $\xi : (u, v) \in [0, 1]^2 \rightarrow \xi_{uv} \in \mathbb{R}$  is a given bounded, measurable function. As usually,  $\langle x, y \rangle$  stands for nearest neighbor vertices.

Let  $\lambda$  be the Lebesgue measure on  $[0, 1]$ . On the set of all configurations on  $A$  the a priori measure  $\lambda_A$  is introduced as the  $|A|$ -fold product of the measure  $\lambda$ . Here and further on  $|A|$  denotes the cardinality of  $A$ . We consider a standard sigma-algebra  $\mathcal{B}$  of subsets of  $\Omega = [0, 1]^V$  generated by the measurable cylinder subsets. A probability measure  $\mu$  on  $(\Omega, \mathcal{B})$  is called a Gibbs measure (with Hamiltonian  $H$ ) if it satisfies the DLR equation, namely for any  $n = 1, 2, \dots$  and  $\sigma_n \in \Omega_{V_n}$ :

$$\mu \left( \left\{ \sigma \in \Omega : \sigma|_{V_n} = \sigma_n \right\} \right) = \int_{\Omega} \mu(d\omega) \nu_{\omega|_{W_{n+1}}}^{V_n}(\sigma_n),$$

where  $\nu_{\omega|_{W_{n+1}}}^{V_n}$  is the conditional Gibbs density

$$\nu_{\omega|_{W_{n+1}}}^{V_n}(\sigma_n) = \frac{1}{Z_n(\omega|_{W_{n+1}})} \exp \left( -\beta H \left( \sigma_n || \omega|_{W_{n+1}} \right) \right),$$

and  $\beta = \frac{1}{T}$ ,  $T > 0$  is temperature. Here and below,  $W_l$  stands for a 'sphere' and  $V_l$  for a 'ball' on the tree, of radius  $l = 1, 2, \dots$ , centered at a fixed vertex  $x^0$  (an origin):

$$W_l = \{x \in V : d(x, x^0) = l\}, \quad V_l = \{x \in V : d(x, x^0) \leq l\};$$

and

$$L_n = \{\langle x, y \rangle \in L : x, y \in V_n\};$$

distance  $d(x, y)$ ,  $x, y \in V$ , is the length of (i.e. the number of edges in) the shortest path connecting  $x$  with  $y$ .  $\Omega_{V_n}$  is the set of configurations in  $V_n$  (and  $\Omega_{W_n}$  that in  $W_n$ ; see below). Furthermore,  $\sigma|_{V_n}$  and  $\omega|_{W_{n+1}}$  denote the restrictions of configurations  $\sigma, \omega \in \Omega$  to  $V_n$  and  $W_{n+1}$ , respectively. Next,  $\sigma_n : x \in V_n \mapsto \sigma_n(x)$  is a configuration in  $V_n$  and  $H(\sigma_n || \omega|_{W_{n+1}})$  is defined as the sum  $H(\sigma_n) + U(\sigma_n, \omega|_{W_{n+1}})$  where

$$H(\sigma_n) = -J \sum_{\langle x, y \rangle \in L_n} \xi_{\sigma_n(x)\sigma_n(y)},$$

$$U(\sigma_n, \omega|_{W_{n+1}}) = -J \sum_{\langle x, y \rangle : x \in V_n, y \in W_{n+1}} \xi_{\sigma_n(x)\omega(y)}.$$

Finally,  $Z_n(\omega|_{W_{n+1}})$  stands for the partition function in  $V_n$ , with the boundary condition  $\omega|_{W_{n+1}}$ :

$$Z_n(\omega|_{W_{n+1}}) = \int_{\Omega_{V_n}} \exp\left(-\beta H(\tilde{\sigma}_n || \omega|_{W_{n+1}})\right) \lambda_{V_n}(d\tilde{\sigma}_n).$$

Due to the nearest-neighbor character of the interaction, the Gibbs measure possesses a natural Markov property: for given a configuration  $\omega_n$  on  $W_n$ , random configurations in  $V_{n-1}$  (i.e., ‘inside’  $W_n$ ) and in  $V \setminus V_{n+1}$  (i.e., ‘outside’  $W_n$ ) are conditionally independent.

We use a standard definition of a translation-invariant measure (see, e.g., [18]). The main object of study in this paper are translation-invariant Gibbs measures for the model (2.1) on Cayley tree. In [17] this problem of description of such measures was reduced to the description of the solutions of a nonlinear integral equation. For finite and countable sets of spin values this argument is well known (see, e.g. [2]- [7], [14], [19], [20], [22]).

Write  $x < y$  if the path from  $x^0$  to  $y$  goes through  $x$ . Call vertex  $y$  a direct successor of  $x$  if  $y > x$  and  $x, y$  are nearest neighbors. Denote by  $S(x)$  the set of direct successors of  $x$ . Observe that any vertex  $x \neq x^0$  has  $k$  direct successors and  $x^0$  has  $k+1$ .

Let  $h : x \in V \mapsto h_x = (h_{t,x}, t \in [0, 1]) \in R^{[0,1]}$  be mapping of  $x \in V \setminus \{x^0\}$  with  $|h_{t,x}| < C$  where  $C$  is a constant which does not depend on  $t$ . Given  $n = 1, 2, \dots$ , consider the probability distribution  $\mu^{(n)}$  on  $\Omega_{V_n}$  defined by

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp\left(-\beta H(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x), x}\right), \quad (2.2)$$

Here, as before,  $\sigma_n : x \in V_n \mapsto \sigma(x)$  and  $Z_n$  is the corresponding partition function:

$$Z_n = \int_{\Omega_{V_n}} \exp\left(-\beta H(\tilde{\sigma}_n) + \sum_{x \in W_n} h_{\tilde{\sigma}(x), x}\right) \lambda_{V_n}(d\tilde{\sigma}_n). \quad (2.3)$$

The probability distributions  $\mu^{(n)}$  are compatible if for any  $n \geq 1$  and  $\sigma_{n-1} \in \Omega_{V_{n-1}}$ :

$$\int_{\Omega_{W_n}} \mu^{(n)}(\sigma_{n-1} \vee \omega_n) \lambda_{W_n}(d(\omega_n)) = \mu^{(n-1)}(\sigma_{n-1}). \quad (2.4)$$

Here  $\sigma_{n-1} \vee \omega_n \in \Omega_{V_n}$  is the concatenation of  $\sigma_{n-1}$  and  $\omega_n$ . In this case there exists a unique measure  $\mu$  on  $\Omega_V$  such that, for any  $n$  and  $\sigma_n \in \Omega_{V_n}$ ,  $\mu\left(\left\{\sigma|_{V_n} = \sigma_n\right\}\right) = \mu^{(n)}(\sigma_n)$ .

**Definition 2.1.** *The measure  $\mu$  is called splitting Gibbs measure corresponding to Hamiltonian (2.1) and function  $x \mapsto h_x$ ,  $x \neq x^0$ .*

The following statement describes conditions on  $h_x$  guaranteeing compatibility of the corresponding distributions  $\mu^{(n)}(\sigma_n)$ .

**Proposition 2.2.** *[17] The probability distributions  $\mu^{(n)}(\sigma_n)$ ,  $n = 1, 2, \dots$ , in (2.2) are compatible iff for any  $x \in V \setminus \{x^0\}$  the following equation holds:*

$$f(t, x) = \prod_{y \in S(x)} \frac{\int_0^1 \exp(J\beta\xi_{tu}) f(u, y) du}{\int_0^1 \exp(J\beta\xi_{0u}) f(u, y) du}. \quad (2.5)$$

Here, and below  $f(t, x) = \exp(h_{t,x} - h_{0,x})$ ,  $t \in [0, 1]$  and  $du = \lambda(du)$  is the Lebesgue measure.

From Proposition 2.2 it follows that for any  $h = \{h_x \in R^{[0,1]}, x \in V\}$  satisfying (2.5) there exists a unique Gibbs measure  $\mu$  and vice versa. However, the analysis of solutions to (2.5) is not easy. This difficulty depends on the given function  $\xi$ . In the next sections we will give a condition on such function under which the corresponding integral equation has unique solution.

### 3. UNIQUENESS OF TRANSLATIONAL - INVARIANT SOLUTION OF (2.5)

In this section we consider  $\xi_{tu}$  as a continuous function and we are going to find a condition on  $\xi_{tu}$  under which the equation (2.5) has unique solution in the class of translational-invariant functions  $f(t, x)$ , i.e.  $f(t, x) = f(t)$ , for any  $x \in V$ . For such functions equation (2.5) can be written as

$$f(t) = \left( \frac{\int_0^1 K(t, u) f(u) du}{\int_0^1 K(0, u) f(u) du} \right)^k, \quad (3.1)$$

where  $K(t, u) = \exp(J\beta\xi_{tu}) > 0$ ,  $f(t) > 0$ ,  $t, u \in [0, 1]$ .

We put

$$C^+[0, 1] = \{f \in C[0, 1] : f(x) \geq 0\}.$$

We are interested to positive continuous solutions to (3.1), i.e. such that

$$f \in C_0^+[0, 1] = \{f \in C[0, 1] : f(x) \geq 0\} \setminus \{\theta \equiv 0\}.$$

Note that equation (3.1) is not linear for any  $k \geq 1$ .

Define the linear operator  $W : C[0, 1] \rightarrow C[0, 1]$  by

$$(Wf)(t) = \int_0^1 K(t, u)f(u)du \quad (3.2)$$

and defined the linear functional  $\omega : C[0, 1] \rightarrow R$  by

$$\omega(f) \equiv (Wf)(0) = \int_0^1 K(0, u)f(u)du. \quad (3.3)$$

Then equation (3.1) can be written as

$$f(t) = (A_k f)(t) = ((Bf)(t))^k, \quad (3.4)$$

where

$$(Bf)(t) = \frac{(Wf)(t)}{(Wf)(0)}, \quad f \in C_0^+[0, 1], \quad k \geq 1. \quad (3.5)$$

**3.1. Existence of solutions to the nonlinear equation (3.4).** In [17] for  $k = 1$  we have proved that the equation (3.4) has unique solution for arbitrary  $K(\cdot, \cdot) \in C^+[0, 1]^2$  and  $f(\cdot) \in C^+[0, 1]$ . But for  $k \geq 2$  the uniqueness is not proved yet. Denote

$$\mathcal{F}_k = \left\{ f \in C^+[0, 1] : f(t) \geq \left( \frac{m}{M_0} \right)^k, k \in \mathbb{N}, \right.$$

where

$$m = \min_{t, u \in [0, 1]} K(t, u), \quad M_0 = \max_{u \in [0, 1]} K(0, u).$$

It is easy to see that  $\mathcal{F}_k$  is a closed and convex subset of  $C[0, 1]$ . Moreover this set is invariant with respect to operator  $A_k$ , i.e.  $A_k(\mathcal{F}_k) \subset \mathcal{F}_k$ .

**Proposition 3.2.** *The operator  $A_k$  is continuous on  $\mathcal{F}_k$  for any  $k \geq 2$ .*

*Proof.* For arbitrary  $C > 0$  we denote

$$\mathcal{F}_0 = \{ f \in C^+[0, 1] : f(t) \geq C, \forall t \in [0, 1] \}.$$

It is obvious that the operator  $A_1$  is continuous on the set  $\mathcal{F}_0$  (see Lemma 2 in [17]).

Let  $f \in \mathcal{F}_k$  be an arbitrary element and  $\{f_n\} \subset \mathcal{F}_k$  such that  $\lim_{n \rightarrow \infty} f_n = f$ . Since the operator  $A_1$  is continuous we have  $\lim_{n \rightarrow \infty} A_1 f_n = A_1 f$ . Consequently, there exists  $C_1 > 0$  such that  $\|A_1 f_n\| \leq C_1$  for  $n \in \mathbb{N}$ . Moreover we have

$$(A_1 f)(t) \leq C_2 = \frac{M}{m_0}, \quad t \in [0, 1],$$

where

$$M = \max_{t, u \in [0, 1]} K(t, u), \quad m_0 = \min_{u \in [0, 1]} K(0, u).$$

We have

$$A_k f_n - A_k f = (Bf_n)^k - (Bf)^k = q_{k,n}(t)(A_1 f_n - A_1 f), \quad (3.6)$$

where

$$q_{k,n}(t) = \sum_{j=0}^{k-1} (A_1 f_n)^{k-j-1}(t) (A_1 f)^j(t) > 0, \quad t \in [0, 1].$$

Consequently,

$$q_{k,n}(t) \leq C = \sum_{j=0}^{k-1} (C_1)^{k-j-1} (C_2)^j, \quad t \in [0, 1].$$

Hence

$$\|A_k f_n - A_k f\| \leq C \|A_1 f_n - A_1 f\|, \quad n \in \mathbb{N}.$$

Since  $A_1$  is a continuous from the last inequality it follows that  $A_k$  is continuous on  $\mathcal{F}_k$ .  $\square$

Denote

$$\mathcal{F}_k^0 = \left\{ f \in C^+[0, 1] : \left( \frac{m}{M_0} \right)^k \leq f(t) \leq \left( \frac{M}{m_0} \right)^k \right\}.$$

**Proposition 3.3.** *Let  $k \geq 2$ . If  $f \in C_0^+[0, 1]$  is a solution of the equation  $A_k f = f$ , then  $f \in \mathcal{F}_k^0$ .*

*Proof.* Straightforward.  $\square$

**Proposition 3.4.** *Let  $k \geq 2$ . The set  $A_k(\mathcal{F}_k^0)$  is relatively compact in  $C[0, 1]$ .*

*Proof.* By Arzelá-Askoli's theorem (see [21], ch.III,§3) it suffices to prove that the set of functions  $A_k(\mathcal{F}_k^0)$  is equi-continuous and there exists  $\gamma > 0$  such that

$$h(t) \leq \gamma, \quad \forall t \in [0, 1] \quad \text{and} \quad \forall h \in A_k(\mathcal{F}_k^0).$$

Let  $h \in A_k(\mathcal{F}_k^0)$  be an arbitrary function, we have

$$0 < h(t) \leq \left( \frac{M}{m_0} \right)^k$$

and there exists a function  $f \in \mathcal{F}_k^0$  such that  $h = A_k f$ .

Now we shall prove that  $A_k(\mathcal{F}_k^0)$  is equi-continuous. For arbitrary  $t, t' \in [0, 1]$  we have ( $h = A_k f$ )

$$\begin{aligned} |h(t) - h(t')| &= |(A_1 f)^k(t) - (A_1 f)^k(t')| = \\ &= \sum_{j=0}^{k-1} (A_1 f)^{k-j-1}(t) (A_1 f)^j(t') |(A_1 f)(t) - (A_1 f)(t')| \leq \\ &\leq k \left( \frac{M}{m_0} \right)^{k-1} \frac{1}{\omega(f)} \int_0^1 |K(t, u) - K(t', u)| f(u) du \leq \\ &\leq k \left( \frac{M}{m_0} \right)^{2k-1} \frac{1}{\omega(f)} \int_0^1 |K(t, u) - K(t', u)| du, \end{aligned}$$

where  $\omega(f)$  is defined in (3.3).

We have

$$\omega(f) \geq m_0 \cdot \left( \frac{m}{M_0} \right)^k, \quad f \in \mathcal{F}_k^0.$$

Consequently,

$$|h(t) - h(t')| \leq \frac{k}{m_0} \left( \frac{M_0}{m} \right)^k \left( \frac{M}{m_0} \right)^{2k-1} \int_0^1 |K(t, u) - K(t', u)| du.$$

Since the kernel  $K(t, u)$  is uniformly continuous on  $[0, 1]^2$ , we conclude that  $A_k(\mathcal{F}_k^0)$  also is equi-continuous.  $\square$

By Propositions 3.2-3.4 and Schauder's theorem (see [13], p.20) one gets the following

**Theorem 3.5.** *The equation  $A_k f = f$  has at least one solution in  $C_0^+[0, 1]$  and the set of all solutions of the equation is a subset in  $\mathcal{F}_k^0$ .*

**3.6. The Hammerstein's nonlinear equation.** For every  $k \in \mathbb{N}$  we consider an integral operator  $H_k$  acting in  $C^+[0, 1]$  as follows:

$$(H_k f)(t) = \int_0^1 K(t, u) f^k(u) du.$$

If  $k \geq 2$  then the operator  $H_k$  is a nonlinear operator which is called Hammerstein's operator of order  $k$ . Moreover the linear operator equation  $H_1 f = f$  has a unique positive solution  $f$  in  $C[0, 1]$  (see [10], p.80).

For a nonlinear homogeneous operator  $A$  it is known that if there is one positive eigenfunction of the operator  $A$  then the number of the positive eigenfunctions is continuum (see [10], p.186).

Denote

$$\mathcal{M}_0 = \{f \in C^+[0, 1] : f(0) = 1\}.$$

**Lemma 3.7.** *The equation*

$$A_k f = f, \quad k \geq 2 \tag{3.7}$$

*has a strongly positive solution iff the equation*

$$H_k f = \lambda f, \quad k \geq 2 \tag{3.8}$$

*has a strongly positive solution in  $\mathcal{M}_0$ .*

*Proof. Necessariness.* Let  $f_0 \in C_0^+[0, 1]$  be a solution of the equation (3.7). We have

$$(W f_0)(t) = \omega(f_0) \sqrt[k]{f_0(t)}.$$

From this equality we get

$$(H_k h)(t) = \lambda_0 h(t),$$

where  $h(t) = \sqrt[k]{f_0(t)}$  and  $\lambda_0 = \omega(f_0) > 0$ .

It is easy to see that  $h \in \mathcal{M}_0$  and  $h(t)$  is an eigenfunction of the Hammerstein's operator  $H_k$ , corresponding the positive eigenvalue  $\lambda_0$ .

*Sufficiency.* Let  $k \geq 2$  and  $h \in \mathcal{M}_0$  be an eigenfunction of the Hammerstein's operator. Then there is a number  $\lambda_0 > 0$  such that  $H_k h = \lambda_0 h$ . From  $h(0) = 1$  we get  $\lambda_0 = (H_k h)(0) = \omega(h^k)$ . Then

$$h(t) = \frac{H_k h}{\omega(h^k)}.$$

From this equality we get  $A_k f_0 = f_0$  with  $f_0 = h^k \in C_0^+[0, 1]$ . This completes the proof.  $\square$

**Theorem 3.8.** *If  $k \geq 2$  then every number  $\lambda > 0$  is an eigenvalue of the Hammerstein's operator  $H_k$ .*

*Proof.* By Theorem 3.5 and Lemma 3.7 there exist  $\lambda_0 > 0$  and  $f_0 \in \mathcal{M}_0$  such that

$$H_k f_0 = \lambda_0 f_0.$$

Take  $\lambda \in (0, +\infty)$ ,  $\lambda \neq \lambda_0$ . Define function  $h_0(t) \in C_0^+[0, 1]$  by

$$h_0(t) = \sqrt[k-1]{\frac{\lambda}{\lambda_0}} f_0(t), \quad t \in [0, 1].$$

Then

$$H_k h_0 = H_k \left( \sqrt[k-1]{\frac{\lambda}{\lambda_0}} f_0 \right) = \lambda h_0.$$

This completes the proof.  $\square$

Denote

$$\mathcal{K} = \left\{ f \in C^+[0, 1] : M \cdot \min_{t \in [0, 1]} f(t) \geq m \cdot \max_{t \in [0, 1]} f(t) \right\},$$

$$\mathcal{P}_k = \left\{ f \in C[0, 1] : \frac{m}{M} \cdot \left( \frac{1}{M} \right)^{\frac{1}{k-1}} \leq f(t) \leq \frac{M}{m} \cdot \left( \frac{1}{m} \right)^{\frac{1}{k-1}} \right\}, \quad k \geq 2.$$

**Proposition 3.9.** *Let  $k \geq 2$ .*

*a) The following holds*

$$H_k(C^+[0, 1]) \subset \mathcal{K}.$$

*b) If a function  $f_0 \in C_0^+[0, 1]$  is a solution of the equation*

$$H_k f = f \tag{3.9}$$

*then  $f_0 \in \mathcal{P}_k$ .*

*Proof.* a) Let  $h \in H_k(C^+[0, 1])$  be an arbitrary function. Then there exists a function  $f \in C^+[0, 1]$  such that  $h = H_k f$ . Since  $h$  is continuous on  $[0, 1]$ , there are  $t_1, t_2 \in [0, 1]$  such that

$$h_{\min} = \min_{t \in [0, 1]} h(t) = h(t_1) = (H_k f)(t_1),$$

$$h_{\max} = \max_{t \in [0, 1]} h(t) = h(t_2) = (H_k f)(t_2).$$



Hence

$$h_{\min} \geq m \int_0^1 f^k(u) du \geq m \int_0^1 \frac{K(t_2, u)}{M} f^k(u) du = \frac{m}{M} h_{\max},$$

i.e.  $h \in \mathcal{K}$ .

b) Let  $f \in C_0^+[0, 1]$  be a solution of the equation (3.9). Then we have  $\|f\| \leq M\|f\|^k$ . Consequently,

$$\|f\| \geq \left(\frac{1}{M}\right)^{\frac{1}{k-1}}.$$

By the property a) we have

$$f(t) \geq f_{\min} = \min_{t \in [0, 1]} f(t) \geq \frac{m}{M} \|f\|.$$

Then we obtain

$$f(t) \geq \frac{m}{M} \left(\frac{1}{M}\right)^{\frac{1}{k-1}}.$$

Also we have

$$f(t) = (H_k f)(t) \geq m \int_0^1 f^k(u) du \geq m f_{\min}^k.$$

Then  $f_{\min} \geq m f_{\min}^k$ , i.e.

$$f_{\min} \leq \left(\frac{1}{m}\right)^{\frac{1}{k-1}}.$$

Hence by the property a) we get

$$f(t) \leq f_{\max} \leq \frac{M}{m} f_{\min} \leq \frac{M}{m} \left(\frac{1}{m}\right)^{\frac{1}{k-1}}.$$

Thus we have  $f \in \mathcal{P}_k$ . □

**3.10. The uniqueness of fixed point of the operators  $A_k$  and  $H_k$ .** Now we shall prove that  $A_k f = f$  and  $H_k f = f$  have a unique solution in  $C_0^+[0, 1]$ .

**Lemma 3.11.** *Assume function  $f \in C[0, 1]$  changes its sign on  $[0, 1]$ . Then for every  $a \in \mathbb{R}$  the following inequality holds*

$$\|f_a\| \geq \frac{1}{n+1} \|f\|, \quad n \in \mathbb{N},$$

where  $f_a = f_a(t) = f(t) - a$ ,  $t \in [0, 1]$ .

*Proof.* By conditions of lemma there are  $t_1, t_2 \in [0, 1]$  such that

$$f_{\min} = f(t_1) < 0, \quad f_{\max} = f(t_2) > 0.$$

In case  $a = 0$  the proof is obvious. We assume  $a > 0$

a) Let  $|f_{\min}| \geq f_{\max}$ . Then  $\|f\| = |f_{\min}| = |f(t_1)|$ . Hence

$$\|f_a\| = \max\{|f(t_1) - a|, |f(t_2) - a|\} = |f(t_1) - a| > |f(t_1)| = \|f\| \geq \frac{1}{n+1} \|f\|, \quad n \in \mathbb{N}.$$

b) Let  $|f_{\min}| < f_{\max}$  and  $\frac{1}{2}\|f\| \geq a$ . Then  $\|f\| = f_{\max} = f(t_2)$  and  $\|f\| - a \geq a > 0$ . Consequently,

$$\|f_a\| = \max\{|f(t_1) - a|, |f(t_2) - a|\} \geq |f(t_2) - a| = \|f\| - a \geq \frac{1}{2}\|f\| \geq \frac{1}{n+1}\|f\|, n \in \mathbb{N}.$$

c) Let  $|f_{\min}| < f_{\max}$  and  $\frac{1}{2}\|f\| < a$ . Then  $\|f\| = f(t_2)$  and

$$\|f_a\| = \max\{|f(t_1) - a|, |f(t_2) - a|\} \geq |f(t_1) - a| > a > \frac{1}{2}\|f\| \geq \frac{1}{n+1}\|f\|, n \in \mathbb{N}.$$

Thus for  $a > 0$  the proof is completed. For  $a < 0$  we put  $g_a(t) = g(t) - a'$  with  $g(t) = -f(t)$  and  $a' = -a > 0$ . Then

$$\|f_a\| = \|g_a\| \geq \frac{1}{n+1}\|g\| = \frac{1}{n+1}\|f\|, n \in \mathbb{N}.$$

This completes the proof.  $\square$

**Theorem 3.12.** *Let  $k \geq 2$ . If the kernel  $K(t, u)$  satisfies the condition*

$$\left(\frac{M}{m}\right)^k - \left(\frac{m}{M}\right)^k < \frac{1}{k}, \quad (3.10)$$

*then the operator  $H_k$  has a unique fixed point in  $C_0^+[0, 1]$ .*

*Proof.* By Theorem 3.8 the Hammerstein's equation  $H_k f = f$  has at least one solution. Assume that there are two solutions  $f_1 \in C_0^+[0, 1]$  and  $f_2 \in C_0^+[0, 1]$ , i.e.  $H_k f_i = f_i$ ,  $i = 1, 2$ . Denote  $f(t) = f_1(t) - f_2(t)$ . Then by Theorem 46.6 of [11] the function  $f(t)$  changes its sign on  $[0, 1]$ . From Lemma 3.11 we get

$$\max_{t \in [0, 1]} \left| f(t) - \frac{k}{2}(\gamma_1 + \gamma_2) \int_0^1 f(s) ds \right| \geq \frac{1}{2}\|f\|,$$

where

$$\gamma_1 = \left(\frac{m}{M}\right)^k, \quad \gamma_2 = \left(\frac{M}{m}\right)^k.$$

By a mean value Theorem we have

$$f(t) = \int_0^1 K(t, u) k \xi^{k-1}(u) f(u) du,$$

here  $\xi \in C^+[0, 1]$  and

$$\min\{f_1(t), f_2(t)\} \leq \xi(t) \leq \max\{f_1(t), f_2(t)\}, t \in [0, 1].$$

By Proposition 3.9 we have  $\xi \in \mathcal{P}_k$ , i.e.

$$\frac{m}{M} \left(\frac{1}{M}\right)^{\frac{1}{k-1}} \leq \xi(t) \leq \frac{M}{m} \left(\frac{1}{m}\right)^{\frac{1}{k-1}}, t \in [0, 1].$$

Hence

$$\gamma_1 \leq K(t, u) \xi^{k-1}(u) \leq \gamma_2, t, u \in [0, 1].$$

Therefore

$$\left| k \cdot K(t, u) \xi^{k-1}(u) - \frac{\gamma_1 + \gamma_2}{2} \right| \leq \frac{\gamma_2 - \gamma_1}{2}.$$

Then

$$\left| f(t) - \frac{k}{2}(\gamma_1 + \gamma_2) \int_0^1 f(u) du \right| \leq \frac{k}{2}(\gamma_2 - \gamma_1) \|f\|. \quad (3.11)$$

Assume the kernel  $K(t, u)$  satisfies the condition (3.10). Then  $k(\gamma_2 - \gamma_1) < 1$  and the inequality (3.11) contradicts to Lemma 3.11. This completes the proof.  $\square$

**Theorem 3.13.** *Let  $k \geq 2$ . If the kernel  $K(t, u)$  satisfies the condition (3.10), then for every  $\lambda > 0$  the Hammerstein's equation  $H_k f = \lambda f$  has unique solution in  $C_0^+[0, 1]$ .*

*Proof.* Clearly the equation  $H_k f = \lambda f$  is equivalent to the following equation

$$\int_0^1 K_\lambda(t, u) f^k(u) du = f(t), \quad (3.12)$$

where  $K_\lambda(t, u) = \frac{1}{\lambda} K(t, u)$ . The kernel  $K_\lambda(t, u)$  satisfies the condition (3.10) with  $\tilde{m} = \frac{m}{\lambda}$  and  $\tilde{M} = \frac{M}{\lambda}$ . Consequently, by Theorem 3.12 it follows that the equation (3.12) has unique solution in  $C_0^+[0, 1]$ .  $\square$

**Theorem 3.14.** *Let  $k \geq 2$ . If the kernel  $K(t, u)$  satisfies the condition (3.10), then the equation  $A_k f = f$  has unique solution in  $C_0^+[0, 1]$ .*

*Proof.* Assume there are two solutions  $f_1, f_2 \in C^+[0, 1]$ ,  $f_1 \neq f_2$ , i.e.  $A_k f_i = f_i$ ,  $i = 1, 2$ . By Lemma 3.7 the functions  $h_i(t) = \sqrt[k]{f_i(t)}$ ,  $t \in [0, 1]$  are solutions of the Hammerstein's equation, i.e.

$$H_k h_i = \lambda_i h_i, \quad i = 1, 2,$$

where  $\lambda_i = \omega(f_i) > 0$  and  $h_i \in \mathcal{M}_0$ . On the other hand Theorem 3.13 implies that  $\lambda_1 \neq \lambda_2$ . Let  $h_0(t) \in C^+[0, 1]$  be a fixed point of the Hammerstein's operator  $H_k$ . Then by Theorems 3.8 and 3.13 we get

$$h_i = \sqrt[k-1]{\lambda_i} h_0(t), \quad i = 1, 2.$$

Consequently,

$$\frac{f_1(t)}{f_2(t)} = \gamma^k, \quad \text{with } \gamma = \sqrt[k-1]{\frac{\lambda_1}{\lambda_2}}.$$

Using this equality we obtain

$$f_1(t) = (A_k f_1)(t) = A_k(\gamma^k f_2) = A_k f_2(t) = f_2(t).$$

This completes the proof.  $\square$

Consider the following Hamiltonian

$$H(\sigma) = -J \sum_{\langle x, y \rangle \in L} \xi_{\sigma(x)\sigma(y)} = - \sum_{\langle x, y \rangle \in L} \ln K(\sigma(x), \sigma(y)), \quad (3.13)$$

where  $J \in \mathbb{R} \setminus \{0\}$  and  $K(t, u)$  satisfies the condition (3.10). Then as a corollary of Proposition 2.2 and Theorem 3.14 we get the following

**Theorem 3.15.** *Let  $k \geq 2$ . If the function  $K(t, u)$  of the Hamiltonian (3.13) satisfies the condition (3.10), then the model (3.13) has unique translational invariant Gibbs measure.*

**Example.** It is easy to see that the condition (3.10) is satisfied iff

$$\frac{M}{m} \leq \eta_k = \sqrt[k]{\frac{1 + \sqrt{4k^2 + 1}}{2k}}, \quad k \geq 2.$$

Consider the following function

$$K(t, u) = \sum_{i=1}^m \sum_{j=1}^n c_{ij} t^i u^j + a, \quad c_{ij} \geq 0, \quad a > 0. \quad (3.14)$$

For this function we have  $m = a$ ,  $M = \sum_{i=1}^m \sum_{j=1}^n c_{ij} + a$ . The following is obvious

- a) If  $\frac{1}{a} \sum_{i=1}^m \sum_{j=1}^n c_{ij} \leq \eta_k - 1$  then for function (3.14) the condition (3.10) is satisfied.
- b) If  $\frac{1}{a} \sum_{i=1}^m \sum_{j=1}^n c_{ij} > \eta_k - 1$  then for function (3.14) the condition (3.10) is not satisfied.

*Remark.* Is there a kernel  $K(t, u) > 0$  of the equation (3.1) when the equation has more than one solutions? This is still open problem.

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YU. KH. ESHKABILOV, NATIONAL UNIVERSITY OF UZBEKISTAN, TASHKENT, UZBEKISTAN.  
*E-mail address:* yusup62@mail.ru

F. H. HAYDAROV, NATIONAL UNIVERSITY OF UZBEKISTAN, TASHKENT, UZBEKISTAN.

U. A. ROZIKOV, INSTITUTE OF MATHEMATICS AND INFORMATION TECHNOLOGIES, TASHKENT, UZBEKISTAN.  
*E-mail address:* rozikovu@yandex.ru